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# Optimal intermittent search strategies

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## Abstract

We study the search kinetics of a single fixed target by a set of searchers performing an intermittent random walk, jumping between different internal states. Exploiting concepts of multi-state and continuous-time random walks we have calculated the survival probability of a target up to time  $t$ , and have ‘optimized’ (minimized) it with regard to the transition probability among internal states. Our model shows that intermittent strategies always improve target detection, even for simple diffusion states of motion.

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## 1. Introduction

Recently, the study of optimal strategies for searching specific targets has experienced a fast growth and motivated a great deal of work. This was due to the subject’s intrinsic relevance in many physical, chemical and biological phenomena, as well as in social sciences and ecology [1–5]. Some recent results deal with the case of the so-called non-destructive search [5–7], in which the target reappears after some time at the same location, or when the searcher always remains inside the system. Others are based on Lévy flights with randomly reoriented ballistic trajectories. All of these provide an optimal search methodology. However another type of random search, an intermittent one, has been recently proposed [1, 3]. Intermittent motion occurs in nature in a wide range of organisms, from protozoans to mammals. It can be found in aerial, aquatic and terrestrial locomotion and in many instinctive behaviors including search and pursuit of prey, mate search, habitat assessment and general travel. Other possibilities include reduced detection or attack by predators and improved endurance [8–10].

Intermittent motion phenomena have been extensively studied for many years in physics and chemistry. Examples of such processes are random walks in dynamically disordered systems [11], such as ionic conduction in polymeric solid electrolytes [12], transport of *Brownions* [13], i.e. particles that can be in two or more states, executing a diffusion process in

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each of them but with different diffusion constants; chromatography, where molecules can be either adsorbed on the column wall or dissolved in the fluid (with transitions from the adsorbed state into the dissolved state at random times), dynamical trapping problems [14, 15], etc. All these phenomena have been analyzed within the framework of the multi-state random walk (MRW) formalism of Weiss and Montroll [16, 17].

Previous works on optimal search strategies [3, 18] were focused on the analysis of the *trapping problem* [1]. A canonical example of this problem is a hunter in search of prey, which are spatially distributed in the field. This problem can be mapped onto the relaxation by the diffusion model [19]. Here we study a related but different situation, the so-called *target problem*, consisting of a single static target and a set of diffusing searchers, initially uniformly distributed over space.

In this work we consider different diffusive motions and the transitions among them. We consider a *continuous-time random walk* (CTRW) with a step size that switches randomly between one and two lattice sites, where the transition rates are exponential functions of time with parameters  $\gamma_1$  and  $\gamma_2$ . This approach differs from those used in previous works [3, 11]. When dealing with the *trapping problem* the average search time should be optimized with respect to the transition rates between the different types of motion. On the other hand, in the *target problem* the survival probability of the target [19, 20] (i.e., the probability that the target remains undetected up to a given time  $t$ ) has to be optimized as a function of the transition rates among the internal states representing different types of motion.

Here we show that intermittent strategies always improve target detection in comparison with the single-state displacement. The main goals of the present paper are two: finding of a non-monotonic dependence of the survival probability on the above-indicated parameters (that can drastically alter the search efficiency), and to complement previous results about optimization of the survival probability of the target [1]. In order to perform our study, we exploit the theory of MRW as well as results from CTRW [16, 17, 21]. We use the concepts of joint conditional probability distribution of a searcher and survival probability of the target, and made the connection to the number of different visited sites by the walker.

The outline of this paper is as follows. In order to make the paper (almost) self-contained, the following section presents a brief review of the MRW [16, 21] and defines the functions to be used later. In section 3, we describe our model, focusing on the one-dimensional case. In section 4, we present some results for the survival probability and related functions, while in the last section we draw some conclusions.

## 2. Multi-state random walk: brief review

We start by briefly reviewing the basic formalism of Montroll and Weiss [16, 17]. We assume that at time  $t$  each walker can be at site  $\vec{s}$  on a lattice, in one of the  $N$  internal states characterized by the following quantities:

- Probability for jump from an internal state:  $f_i(t) dt$  is the probability of leaving state  $i$  between  $t$  and  $t + dt$ , given that the walker arrived to that state at  $t = 0$ ;
- Probability of staying in a given state:  $F_i(t) = \int_t^\infty f_i(t) dt$  is the probability of remaining with state  $i$ , the sojourn  $t$ , given that the walker arrived to it at time  $t = 0$ ;
- *Propagator*:  $\xi_i(\vec{s}, \vec{s}'; t)$  is the probability of finding the walker at site  $\vec{s}$ , in state  $i$ , and at time  $t$ , given that it was at  $\vec{s}'$  in state  $i$  at  $t = 0$ , and without changing its internal state during the motion.

We also define the following auxiliary functions:

$$h_i(\vec{s}, \vec{s}'; t) = \xi_i(\vec{s}, \vec{s}'; t) f_i(t), \quad H_i(\vec{s}, \vec{s}'; t) = \xi_i(\vec{s}, \vec{s}'; t) F_i(t).$$

We assume that at  $t = 0$  the walker is at  $\vec{s} = \vec{0}$ , with a probability  $g_i$  of being in state  $i$ . Particularly, in this work we consider two internal states ( $N = 2$ ), and assume translational invariance in each state of motion, i.e.  $\xi_i(\vec{s}, \vec{s}'; t) = \xi_i(\vec{s} - \vec{s}'; t)$ .

Now, we denote by  $P_j(\vec{s}, t | \vec{0}, t = 0)$  the joint conditional probability of being at site  $\vec{s}$  in the internal state  $j$  ( $j = 1, 2$ ), independently of the initial internal state. The expressions for these probabilities (in the Fourier–Laplace space) are given by [16]

$$\hat{P}_1(\vec{k}, u | \vec{0}, t = 0) = \frac{g_1 \hat{H}_1(\vec{k}; u) + g_2 \hat{H}_1(\vec{k}; u) \hat{h}_2(\vec{k}; u)}{1 - \hat{h}_1(\vec{k}; u) \hat{h}_2(\vec{k}; u)}, \quad (1)$$

$$\hat{P}_2(\vec{k}, u | \vec{0}, t = 0) = \frac{g_2 \hat{H}_2(\vec{k}; u) + g_1 \hat{H}_2(\vec{k}; u) \hat{h}_1(\vec{k}; u)}{1 - \hat{h}_1(\vec{k}; u) \hat{h}_2(\vec{k}; u)}. \quad (2)$$

The reported results are general, valid for any probability density  $f_j(t)$ , as well as for any propagator  $\xi_j(\vec{s} - \vec{s}'; t)$ , within each state.

In the following, we consider two important particular cases for the transitions between internal states and the corresponding propagators:

(a) Markovian transitions between internal states

Assuming that the transition between the internal states is Poissonian, i.e.

$$f_j(t) = \gamma_j e^{-\gamma_j t}, \quad (3)$$

where  $j = 1, 2$ , it can be shown that

$$\hat{h}_i(\vec{k}, u) = \gamma_i \hat{\xi}_i(\vec{k}; u + \gamma_i), \quad (4)$$

$$\hat{H}_i(\vec{k}, u) = \hat{\xi}_i(\vec{k}; u + \gamma_i). \quad (5)$$

Then,

$$\hat{P}_1(\vec{k}, u | \vec{0}, t = 0) = \frac{g_1 \hat{\xi}_1(\vec{k}; u + \gamma_1) + g_2 \hat{\xi}_1(\vec{k}; u + \gamma_1) \gamma_2 \hat{\xi}_2(\vec{k}; u + \gamma_2)}{1 - \gamma_1 \hat{\xi}_1(\vec{k}; u + \gamma_1) \gamma_2 \hat{\xi}_2(\vec{k}; u + \gamma_2)}, \quad (6)$$

$$\hat{P}_2(\vec{k}, u | \vec{0}, t = 0) = \frac{g_2 \hat{\xi}_2(\vec{k}; u + \gamma_2) + g_1 \hat{\xi}_1(\vec{k}; u + \gamma_2) \gamma_1 \hat{\xi}_2(\vec{k}; u + \gamma_1)}{1 - \gamma_1 \hat{\xi}_1(\vec{k}; u + \gamma_1) \gamma_2 \hat{\xi}_2(\vec{k}; u + \gamma_2)}. \quad (7)$$

These expressions can also be recast as

$$\hat{P}_1(\vec{k}, u | \vec{0}, t = 0) = g_1 \hat{P}_{11}(\vec{k}, u | \vec{0}, t = 0) + g_2 \hat{P}_{12}(\vec{k}, u | \vec{0}, t = 0), \quad (8)$$

$$\hat{P}_2(\vec{k}, u | \vec{0}, t = 0) = g_1 \hat{P}_{21}(\vec{k}, u | \vec{0}, t = 0) + g_2 \hat{P}_{22}(\vec{k}, u | \vec{0}, t = 0), \quad (9)$$

where  $P_{ij}(\vec{s}, t | \vec{0}, t = 0)$  is the joint conditional probability of being at site  $\vec{s}$  in the internal state  $i$  at time  $t$ , given that at  $t = 0$  it was at  $\vec{s} = \vec{0}$  in internal state  $j$ .

(b) CTRW propagators

For CTRW, the form of the propagators (in Fourier–Laplace space) is

$$\hat{\xi}_j(\vec{k}; u) = \frac{1 - \hat{\Psi}_j(\vec{k} = \vec{0}; u)}{u[1 - \hat{\Psi}_j(\vec{k}; u)]}, \quad (10)$$

with  $j = 1, 2$  and where  $\Psi_j(\vec{r}, t)$  is the probability that the walker makes a transition to site  $\vec{r}$  between  $t$  and  $t + dt$  given that it arrived at site  $0$  at  $t = 0$  without changing its internal state. Moreover, if the functions  $\Psi_j(\vec{r}, t)$  are separable [16, 17], we have  $\hat{\Psi}_j(\vec{k}; u) = \psi_j(u) p_j(\vec{k})$ . Here  $\psi_j(u)$  is the Laplace transform of the waiting time density function and  $p_j(\vec{k})$  is the structure factor of the lattice.

### 3. Model for diffusive intermittent searching

We restrict our work to an infinite discrete chain. However, this scheme can also be applied to higher dimensions and in finite size domains. We assume that there is only one *fixed target/prey* located at the origin of the lattice and a *set of searchers/predators*, initially uniformly distributed along the chain. We denote the initial occupation probability of each site by  $\rho$ . The ‘search’ starts at  $t = 0$  and when the first searcher finds the target, captures it with probability one, and the ‘game is over’.

Following [1], we assume that each predator can make its diffusive motion on the lattice with two types of displacements, jumping between both states. In one of these states, the searcher explores different regions of the lattice, and in the other state, the predator makes a compact exploration of its local area. An intermittent strategy consists of choosing the rates of transition between these internal states. The reason why an intermittent strategy may be more efficient than simple diffusion is that long jumps give the possibility to go away from a well-explored region and falling down in the target’s neighborhood. So a combination of both types of displacements optimize the detection process. We want to show the effectiveness of the intermittency using the simplest case for diffusive motion in a chain:

- *state-1*, corresponding to a RW with symmetric jumps to second neighbors (next-nearest neighbors), with a frequency  $\lambda_1$ ;
- *state-2*, also corresponding to a RW with symmetric jumps, but to first neighbors, with a frequency  $\lambda_2$ .

Let us assume that we have a separable RW. The Laplace transform of the waiting time functions in the Poisson case are  $\psi_j(u) = \lambda_j/\lambda_j + u$ , and  $f_j(t)$  are given by equations (3). In addition, the structure factors in one dimension result  $p_1(k) = \cos(2k)$  and  $p_2(k) = \cos(k)$ .

The optimization problem consists of finding the values of the  $\gamma_j$  which make the prey’s *survival probability*, at a given time  $t$ , be *minimal*. The survival probability [1, 16, 17, 21],  $\phi(t)$ , could be written as  $\phi(t) = e^{-\rho S(t)}$ , where  $S(t)$  corresponds to the number of different sites visited in any of the two internal states until time  $t$ . It could be shown [21] that for the target problem in any of the two internal states,

$$\tilde{S}_{\text{Mar}}(u) = \frac{1}{u^2 \hat{P}_{\text{Mar}}(s=0, u|0, t=0)}. \quad (11)$$

$P_{\text{Mar}}(s, t|0, t=0)$  must be interpreted as the marginal probability for finding the predator at site  $s$  at time  $t$ , independently of the internal state; i.e.,  $P_{\text{Mar}}(s, t|0, t=0) = P_1(s, t|0, t=0) + P_2(s, t|0, t=0)$ . If the target detection only occurs in one of the two internal states (here, we assume that the ‘capture’ is only possible in *state-2*), the survival probability  $\phi_2(t)$  could be written as  $\phi_2(t) = e^{-\rho S_2(t)}$ , where

$$\tilde{S}_2(u) = \frac{1}{u} \frac{g_1 \hat{P}_{21}(k=0, u|0, t=0) + g_2 \hat{P}_{22}(k=0, u|0, t=0)}{\tilde{P}_{22}(s=0, u|0, t=0)}. \quad (12)$$

As usual, we evaluate  $\tilde{P}_{ij}(s, u|0, t=0)$  through the inverse transformation

$$\tilde{P}_{ij}(s, u|0, t=0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{P}_{ij}(k, u|0, t=0) e^{iks} dk. \quad (13)$$

In the following lines, and in order to simplify, we assume  $\lambda_1 = \lambda_2 = \lambda$ , which allows us to scale the  $\gamma_j$  in this parameter.

Now, we consider two important cases that can be solved analytically. Firstly, the regime of high transition rate between internal states; and secondly, the regime without transitions between internal states.

(a) Regime of high transition rate between internal states

If  $(\gamma_1 + \gamma_2) \gg \lambda$  and  $u \ll (\gamma_1 + \gamma_2)$ ,  $\hat{P}_{\text{Mar}}(s, u|0, t = 0)$  may be expressed as

$$\tilde{P}_{\text{Mar}}(s, u|0, t = 0) = \frac{1}{\pi} \int_0^\pi \frac{\cos(ks)}{(1+u) - \left(\frac{\gamma_1}{\gamma_1 + \gamma_2}\right)\cos(k) - \left(1 - \frac{\gamma_1}{\gamma_1 + \gamma_2}\right)\cos(2k)} dk, \quad (14)$$

where the  $\gamma_i$  are measured in units of  $\lambda$ . Calling  $\alpha = \frac{\gamma_1}{\gamma_1 + \gamma_2}$  and  $z = \frac{1}{u+1}$  we can write

$$\hat{P}_{\text{Mar}}(s, u|0, t = 0) = \frac{z}{\pi} \int_0^\pi \frac{\cos(ks)}{1 - z[\alpha \cos(k) + (1 - \alpha)\cos(2k)]} dk. \quad (15)$$

This expression may be compared with equation (4.4) in [1]. Using a Tauberian theorem [22], it can be shown that in this case

$$S(t; \alpha) \sim \frac{2\sqrt{2(4-3\alpha)}}{\sqrt{\pi}} t^{1/2} - \frac{4(1-\alpha)}{\sqrt{\alpha(4-3\alpha)}} \quad (\alpha \neq 0, t \gg 1). \quad (16)$$

In the following section, we will show the behavior of  $S(t; \alpha)$  as a function of  $\alpha$  for a given fixed time  $t$ .

(b) Regime of non-transitions between internal states: single-state search

In the *single-state search* ( $\gamma_1 = \gamma_2 = 0$ )

$$S_{\text{single}}(u) = \frac{1}{u^2 P_{11}(0, u|0, t = 0)}. \quad (17)$$

The corresponding expression in the time domain obtained from equation (17) results

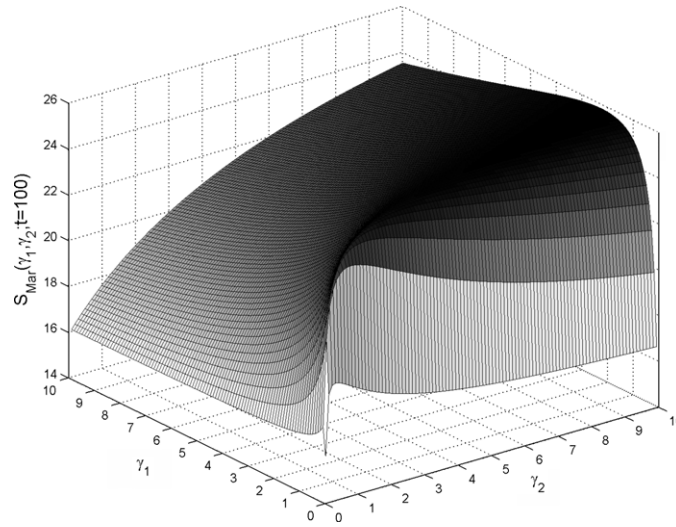
$$S_{\text{single}}(t) = 1 + \int_0^t e^{-t'} (I_0(t') + I_1(t')) dt' \quad (18)$$

where  $I_v(t)$  is the first modified Bessel function of order  $v$ . It can be also shown that  $\tilde{P}_{11}(s = 0, u|0, t = 0) = \tilde{P}_{22}(s = 0, u|0, t = 0)$ , and hence,  $\tilde{S}_{\text{single}}(u)$  is the same for nearest and next-nearest neighbor transitions.

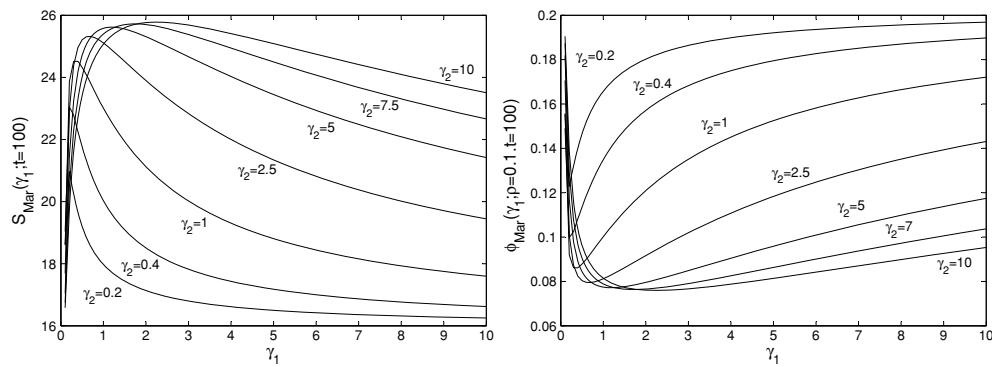
#### 4. Numerical results

In this section, we illustrate the results given by equations (11) and (12), using the numerical procedure of [23] for calculating the inverse Laplace transform. Also, we give some general ideas to interpret these results. For simplicity, we choose that both internal states have initial equal probabilities:  $g_1 = g_2 = 1/2$ , and time is given in units of the diffusion constant ( $\lambda$ ). We also choose for the concentration of searchers to be  $\rho = 0.1$ .

In figure 1, we draw the surface  $S_{\text{Mar}}(\gamma_1, \gamma_2; t)$  for a fixed time ( $t = 100$ ). Note how the intermittent search can improve (maximize) the number of distinct sites visited, compared with the *single-state search* ( $\gamma_1 = \gamma_2 = 0$ ). From the figure it is apparent that a region of optimal values in the parameter space ( $\gamma_1, \gamma_2$ ) arises, that can be appreciated by the gray scale, where darker means a larger value in  $S_{\text{Mar}}(\gamma_1, \gamma_2; t)$ . We want to stress that, for a given value of  $\gamma_1$  (given a fixed value of  $t$ ), we can always choose  $\gamma_2$  in such a way to make  $S_{\text{Mar}}(\gamma_1, \gamma_2; t)$  maximum (and vice versa). The curves shown in figure 2 correspond to cuts of the surface indicated in figure 1, with  $\gamma_2 = cte$ , for  $t = 100$ . We also show sections corresponding to the survival probability  $\phi_{\text{Mar}}(\gamma_1, \gamma_2; t)$ . Here we can see that, for a fixed value of  $\gamma_2$ , we always found a minimum as a function of  $\gamma_1$ , i.e. we can optimize the search at the given value of  $t$ .



**Figure 1.**  $S_{\text{Mar}}(\gamma_1, \gamma_2; t = 100)$ , the number of distinct sites visited in the marginal case, i.e. capture in any of the states, as a function of the transition rates  $\gamma_1$  and  $\gamma_2$ .

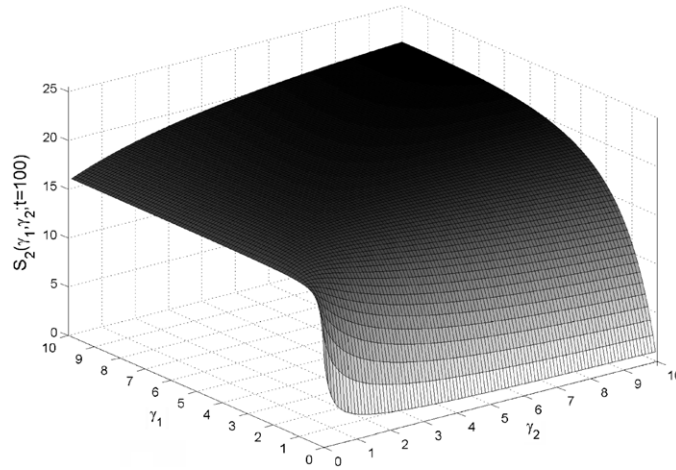


**Figure 2.** Left panel: sections of  $S_{\text{Mar}}(\gamma_1, \gamma_2 = \text{cte}; t)$  versus  $\gamma_1$ . These plots correspond to cuts of the surface sketched in figure 1. Right panel: plots of the corresponding survival probability for each value of  $\gamma_2$  selected in the left panel.

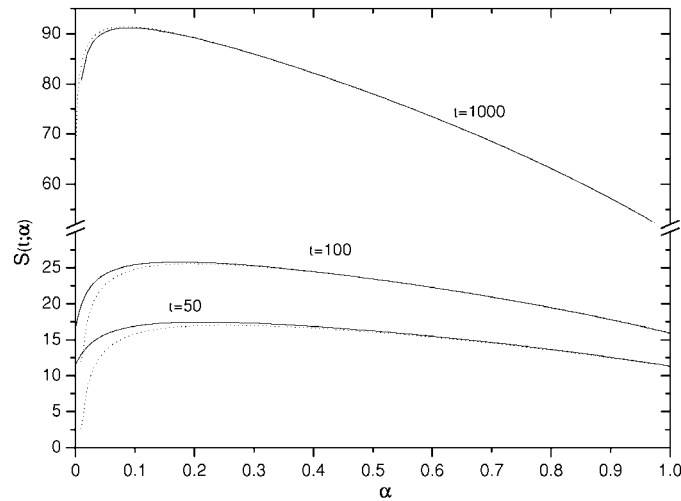
Moreover, we get an important reduction for the survival probability ( $\sim 50\%$ ) for an optimal choice of the transition rates.

Figure 3 depicts the behavior of  $S_2(\gamma_1, \gamma_2; t)$  for search with capture only in *state-2*. It shows that we still can obtain a detection's improvement setting the proper values of  $(\gamma_1, \gamma_2)$ . Remembering that  $T_1 = 1/\gamma_1$  is the mean stay time in the single-state search, the limit  $\gamma_1 \rightarrow 0$  shows the expected behavior,  $S_2(\gamma_1, \gamma_2; t) \rightarrow 0$ .

Figure 4 corresponds to the asymptotic limit of high transition rates between internal states. It depicts the curves obtained for three different times:  $t = 50, 100, 1000$ , as a function of the parameter  $\alpha$ . Note how the searcher, in order to maximize the number of distinct sites visited, must spend more time in the re-localization state. The value of  $\alpha$  that maximizes  $S(t; \alpha)$ , for a fixed  $t$ , defines the optimal relation between  $\gamma_1$  and  $\gamma_2$  for improving the search strategy.



**Figure 3.** The number of distinct sites visited,  $S_2(\gamma_1, \gamma_2; t = 100)$ , as a function of  $\gamma_1$  and  $\gamma_2$ , when target detection takes place only in the internal *state-2*.



**Figure 4.** Plot of  $S(t; \alpha)$  versus  $\alpha$  for the fixed values of times  $t = 50, 100, 1000$ . The analytical results given in equation (16) are plotted in dots for comparison.

### 5. Conclusions

We have presented a simple model for the search kinetics of a hidden target located at the origin of a chain, by a set of searchers performing intermittent diffusive motion. Each searcher can be at a particular site on the lattice in one of the two internal states. While a walker is in internal *state-1* (2), it performs a random walk with symmetric jumps to second (first) neighbors with diffusion constant  $\lambda$  (in both cases). The intermittency is characterized by the probability density functions for jumps between the internal states. For simplicity, we have restricted the analysis of intermittency to time exponential functions with parameters  $\gamma_1$  and  $\gamma_2$ . However, our framework is completely general.



We evaluated the target's survival probability at a fixed time as a function of the parameters  $\gamma_1$  and  $\gamma_2$  when the capture of the prey (or detection of the target) only occurs in one or both internal states. We have established that the survival is a non-monotonic function of  $\gamma_1$  and  $\gamma_2$ , and that for a given value of one of these parameters, there exists a corresponding value of the other such that the search efficiency can be drastically enhanced. Furthermore, intermittent strategies always improve target detection when compared with the single-state displacement.

Thus, we have fulfilled the goal of presenting a simple model based only on diffusion that captures the main features of intermittent search which are relevant for optimization strategies. We also have reinterpreted and complemented previous related results given in [1]. Our model of intermittent search can be straightforwardly generalized to higher dimensions, finite size domain systems, continuous systems, 'imperfect' detection, 'dynamical' behavior of the target, non-Markovian transitions between internal states, etc. All these aspects will be the subject of further works.

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